M 317 Exam 2 Continuity

Take Home part of the exam: (**20 pts each**)

1. Suppose $f \in C[0,10]$ and let $\{b_n\}$ denote any sequence in $rng[f]$. Show (without appealing to the compact range theorem) that $\{b_n\}$ contains a convergent subsequence.

let $\{b_n\}$ denote any sequence in $rng[f]$, then $\exists \{a_n\}$ a sequence in $[0,10]$
such that $f(a_n) = b_n \ \forall n$ (definition of $rng[f]$)
Since $\{a_n\} \subset [0,10]$, a compact set, $\exists$ subsequence $\{a_{n'}\}$ converging to A in $[0,10]$
Since $f \in C[0,10], f(a_{n'})$ converges to $f(A)$
Then $\{b_{n'}\} = \{f(a_{n'})\}$ is a convergent subsequence of $\{b_n\}$

2. Show that $f(x) = 1 + x^2 - 10\sin(x)$ has at least one zero in $(0,\infty)$.

$f(x) \in C(R)$ so the intermediate value theorem applies
$f(0) = 1 > 0$ and $f(\pi/2) = 1 + (\pi/2)^2 - 10 = -6.5 < 0$
since $f(0) > 0 > f(\pi/2)$, the IVT implies f has a zero between 1 and $\pi/2$

3. The function $f(x) = \sin x \cos x$ is continuous on $(-\infty,\infty)$. Is it uniformly continuous?

$f(x) = \frac{1}{2}\sin 2x$ is continous on $(-\pi, 2\pi)$ so it is certainly continuous on $[0,\pi]$
then uniformly continuous on $[0,\pi]$ since $[0,\pi]$ is compact.
since $f(x) = f(x+\pi)$ for all x, f is uniformly continuous on $[k\pi,(k+1)\pi]$ for all k
Then f is uniformly continuous for all x.

4. Is $f(x) = \frac{\sin^2 x}{x\sqrt{x}}$ uniformly continuous on $(0,1)$?

Clearly f is continuous on $(0,1]$ by arithmetic for continuous functions
$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin^2 x}{x^2}\sqrt{x} = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2 \sqrt{x} = 1 \cdot 0$ using arithmetic with limits
then if we define $f(0) = 0$, f is continuous on $[0,1]$ and since $[0,1]$ is compact,
f is uniformly continuous on $[0,1]$ and on $(0,1)$

5. For the extreme value theorem, and for the intermediate value theorem, give an example where one of the hypotheses of the theorem is NOT satisfied and the conclusion of the theorem fails. Be sure to indicate which hypothesis is not satisfied for your examples.

the extreme value theorem- If f is continous on a compact domain, then
there exist points c,d in $dom[f]$ where $f(c) = \inf_D f$ and $f(d) = \sup_D f$.
$f(x) = 1/x$ on $dom[f] = (0,\infty)$ assumes neither the $\sup$ nor the $\inf$ at a point of the domain here the domain is not compact. If we have $dom[f] = [0,1]$, then the domain is compact but f is not continuous on D.

the intermediate value theorem- If f is continous on an interval, $D,$ and $F_1 < F_2$ belong to $rng[f]$ with $F_1 < c < F_2,$ then there exists a point p in D where $f(p) = c.$

$f(x) = 1/x$ is continuous on $D = (-1,0) \cup (0,1).$
But D is not an interval, so there is no p in D where $f(p) = 0$

If $f(x) = \begin{cases} |x|/x & if \quad x \neq 0 \\ 1 & if \quad x = 0 \end{cases}$ then $-1 = F_1 < 0 < F_2 = 1$ but there is no p where $f(p) = 0.$ This f is not continous on $D = R$

6. Suppose $|f(x)| \leq M$ for all x in $(0,1)$ and $\lim_{x \to 0} f(x)$ fails to exist.
Show that there must be sequences $\{a_n\}$ and $\{b_n\}$ in $(0,1)$ such that $a_n \to 0$ and $b_n \to 0$ while $\{f(a_n)\}$ and $\{f(b_n)\}$ both converge but to different limits.

Since $\lim_{x \to 0} f(x)$ fails to exist, f is not continuous at $x = 0$ (x=0 is not a removable disc)
Then f has one of three types of discontinuity at $x = 0$: infinite, oscillatory or jump.
Since $|f(x)| \leq M$ for all x in $(0,1)$, the discontinuity is not infinite.
the discontinuity can't be a jump since $x = 0$ is an endpoint of the domain which means x can't approach 0 from the left.
Finally, if f has an oscillatory discontinuity at $x = 0,$ then we can find sequences $\{a_n\}, \{b_n\}$ in $(0,1)$ such that $a_n \to 0$ and $b_n \to 0$ while $\{f(a_n)\}$ and $\{f(b_n)\}$ both converge but to different limits.